

A POINTWISE BOUND FOR A HOLOMORPHIC FUNCTION WHICH IS SQUARE-INTEGRABLE WITH RESPECT TO AN EXPONENTIAL DENSITY FUNCTION

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ABSTRACT. Let φ be a real-valued smooth function on \mathbb{C} satisfying $0 \leq \Delta\varphi \leq M$ for some $M \geq 0$. Denote by $\mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$ the space of all holomorphic functions which are square-integrable with respect to the measure $e^{-\varphi(z)} dz$. In this paper, we obtain a pointwise bound for any function in this space. We show that there exists a constant K depending only on M such that

$$|f(z)|^2 \leq K e^{\varphi(z)} \|f\|_{L^2(\mathbb{C}, e^{-\varphi})}^2$$

for any $f \in \mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$ and any $z \in \mathbb{C}$.

1. INTRODUCTION

Let U be a non-empty open subset of \mathbb{C} . Denote by $\mathcal{HL}^2(U, \alpha)$ the space of all holomorphic functions on U which are square-integrable with respect to the measure $\alpha(\omega) d\omega$.

For any $t > 0$, consider the Gaussian measure

$$d\mu_t(z) = \frac{1}{\pi t} e^{-|z|^2/t} dz.$$

Then the space $\mathcal{HL}^2(\mathbb{C}, \mu_t)$ is called the *Segal-Bargmann space*. See [GM], [H1], [H2], [F] for detailed discussion about the importance of this space, and its relevance in quantum theory. It is well-known that a pointwise bound for any function $f \in \mathcal{HL}^2(\mathbb{C}, \mu_t)$ is given by

$$(1.1) \quad |f(z)|^2 \leq e^{|z|^2/t} \|f\|_{L^2(\mathbb{C}, \mu_t)}^2.$$

This pointwise bound first appeared in Bargmann's paper [B] and was revisited many times by other authors. More generally, for any space $\mathcal{HL}^2(U, \alpha)$, there exists a function $K(z, \omega)$ on $U \times U$, called the *reproducing kernel*, such that

$$(1.2) \quad |f(z)|^2 \leq K(z, z) \|f\|_{L^2(U, \alpha)}^2$$

for any $f \in \mathcal{HL}^2(U, \alpha)$ and $z \in U$. The Bargmann's pointwise bound (1.1) for $\mathcal{HL}^2(\mathbb{C}, \mu_t)$ follows from the following formula of the reproducing kernel

for the Segal-Bargmann space:

$$(1.3) \quad K(z, \omega) = e^{z\bar{\omega}/t}.$$

In this work, we study a pointwise bound for a function in a more general holomorphic function space. First, we look at the space $\mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$, where $\Delta\varphi$ is a positive constant. Note that $\Delta(|z|^2/t) = 4/t > 0$, so this is a generalization of the standard Segal-Bargmann space $\mathcal{HL}^2(\mathbb{C}, \mu_t)$. The technique used here will be that of holomorphic equivalence [H1]. Two holomorphic function spaces $\mathcal{HL}^2(U, \alpha)$ and $\mathcal{HL}^2(U, \beta)$ are holomorphically equivalent if there exists a nowhere-zero holomorphic function ϕ on U such that

$$\beta(z) = \frac{\alpha(z)}{|\phi(z)|^2} \quad \text{for all } z \in U.$$

If $\mathcal{HL}^2(U, \alpha)$ and $\mathcal{HL}^2(U, \beta)$ are holomorphically equivalent spaces, then their reproducing kernels are related by

$$(1.4) \quad \alpha(z)K_\alpha(z, z) = \beta(z)K_\beta(z, z).$$

We show that if $\Delta\varphi = c > 0$, then $\mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$ is holomorphically equivalent to the Segal-Bargmann space $\mathcal{HL}^2(\mathbb{C}, \mu_t)$ where $t = 4/c$. It follows from (1.2) and (1.4) that

$$|f(z)|^2 \leq \frac{c}{4\pi} e^{\varphi(z)} \|f\|_{L^2(\mathbb{C}, e^{-\varphi})}^2,$$

for any $f \in \mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$ and any $z \in \mathbb{C}$.

Next, we turn to the space $\mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$, where $\Delta\varphi$ is positive and bounded, i.e. $0 \leq \Delta\varphi \leq M$ for some $M \geq 0$. This space is not holomorphically equivalent to a Segal-Bargmann space, so we cannot apply the same technique here. Our proof relies on a technical lemma which can be stated as follows: For any $f \in \mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$,

$$|f(0)|^2 \leq C e^{\varphi(0)} \int_{D(0,1)} |f(\omega)|^2 e^{-\varphi(\omega)} d\omega$$

for some C depending only on M . By translation to any point $z \in \mathbb{C}$, we obtain the following pointwise bound:

$$|f(z)|^2 \leq C e^{\varphi(z)} \|f\|_{L^2(\mathbb{C}, e^{-\varphi})}^2.$$

Here is a brief summary of this work. In section 2, we study basic properties of holomorphic function spaces. We introduce the concept of holomorphic equivalence and establish a necessary and sufficient condition for two spaces to be holomorphically equivalent. In section 3, we establish a pointwise bound for functions in $\mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$.

2. HOLOMORPHIC FUNCTION SPACES

In this section, we review and prove some relevant facts about holomorphic function spaces that are needed in this paper. The main reference here is [H1].

Let U be a non-empty open subset of \mathbb{C} . Denote by $\mathcal{H}(U)$ the space of all holomorphic functions on U . If α is a strictly positive function on U , let $L^2(U, \alpha)$ be the space of all functions on U which are square-integrable with respect to the measure $\alpha(\omega) d\omega$. Then $L^2(U, \alpha)$ is a Hilbert space. Let $\mathcal{H}L^2(U, \alpha) = \mathcal{H}(U) \cap L^2(U, \alpha)$. Then $\mathcal{H}L^2(U, \alpha)$ is a closed subspace of $L^2(U, \alpha)$ and hence a Hilbert space. Moreover, it is well-known that $\mathcal{H}L^2(U, \alpha)$ is separable.

Definition 1. A *Segal-Bargmann space* is a space $\mathcal{H}L^2(\mathbb{C}, \mu_t)$, where

$$\mu_t(z) = \frac{1}{\pi t} e^{-|z|^2/t}$$

for some $t > 0$.

Let $K: U \times U \rightarrow \mathbb{C}$ be a reproducing kernel for the space $\mathcal{H}L^2(U, \alpha)$. We refer to [H1] for details of the discussion below. If $\{e_i\}_{i=0}^\infty$ is an orthonormal basis for $\mathcal{H}L^2(U, \alpha)$, then the reproducing kernel K is given by

$$(2.1) \quad K(z, \omega) = \sum_{i=0}^{\infty} e_i(z) \overline{e_i(\omega)} \quad (z, \omega \in U).$$

If we know the reproducing kernel of the space, the pointwise bound of any function f in $\mathcal{H}L^2(U, \alpha)$ can be obtained by

$$(2.2) \quad |f(z)|^2 \leq K(z, z) \|f\|_{L^2(U, \alpha)}^2.$$

Moreover, for a fixed value of z , $K(z, z)$ is the smallest constant which makes the pointwise bound (2.2) holds for all $f \in \mathcal{H}L^2(U, \alpha)$.

Definition 2. Holomorphic function spaces $\mathcal{H}L^2(U, \alpha)$ and $\mathcal{H}L^2(U, \beta)$ are said to be *holomorphically equivalent* spaces if there exists a nowhere zero holomorphic function ϕ on U such that

$$\beta(z) = \frac{\alpha(z)}{|\phi(z)|^2} \quad \text{for all } z \in U.$$

In this case, the map $f \mapsto \phi f$ is a unitary map from $\mathcal{H}L^2(U, \alpha)$ onto $\mathcal{H}L^2(U, \beta)$.

Lemma 3. Let $\mathcal{H}L^2(U, \alpha)$ and $\mathcal{H}L^2(U, \beta)$ be holomorphically equivalent spaces. Let K_α and K_β be their respective reproducing kernels. Then for each $z \in U$,

$$\alpha(z) K_\alpha(z, z) = \beta(z) K_\beta(z, z).$$

Proof. By formula 2.1 and the fact that a unitary map preserves orthonormal bases, we obtain

$$K_\beta(z, \omega) = \phi(z) \overline{\phi(\omega)} K_\alpha(z, \omega).$$

It follows that

$$K_\beta(z, z) = |\phi(z)|^2 K_\alpha(z, z) = \frac{\alpha(z)}{\beta(z)} K_\alpha(z, z).$$

Thus, $\alpha(z) K_\alpha(z, z) = \beta(z) K_\beta(z, z)$. □

The next goal in this section is to establish a necessary and sufficient condition for two spaces to be holomorphically equivalent.

Lemma 4. *Let U be an open simply connected set in \mathbb{C} and α a strictly positive smooth function on U . Then there exists a holomorphic function ϕ such that $|\phi|^2 = \alpha$ if and only if $\log \alpha$ is harmonic.*

Proof. (\Rightarrow) Since $\phi \in \mathcal{H}(U)$, by a standard result in complex analysis, there exists a function $\theta \in \mathcal{H}(U)$ such that $\phi = e^\theta$. Let $u = \operatorname{Re} \theta$. Thus, $|\phi| = e^u$ and hence $\alpha = e^{2u}$. Then $\log \alpha = 2u$, which implies that $\Delta \log \alpha = \Delta 2u = 0$. (\Leftarrow) Assume that $u = \log \alpha$ is harmonic. Then there exists a holomorphic function f such that $u = \operatorname{Re} f$. Hence, e^f is also holomorphic. Let $\phi = e^{f/2}$. Then $\phi \in \mathcal{H}(U)$ and $e^f = \phi^2$. Hence, $\alpha = e^u = |e^f| = |\phi|^2$. \square

Proposition 5. *Let U be an open simply connected set in \mathbb{C} and α, β strictly positive smooth functions on U . Then $\mathcal{HL}^2(U, \alpha)$ and $\mathcal{HL}^2(U, \beta)$ are holomorphically equivalent spaces if and only if $\Delta \log \alpha(z) = \Delta \log \beta(z)$.*

Proof. If $\mathcal{HL}^2(U, \alpha)$ and $\mathcal{HL}^2(U, \beta)$ are holomorphically equivalent, then there is a function $\phi \in \mathcal{H}(U)$ such that $\phi \neq 0$ and $|\phi(z)|^2 = \frac{\alpha(z)}{\beta(z)}$. By Lemma 4, $\log \frac{\alpha(z)}{\beta(z)}$ is harmonic. Hence, $\Delta(\log \alpha(z) - \log \beta(z)) = 0$, which shows that $\Delta \log \alpha(z) = \Delta \log \beta(z)$. It is easy to see that the reverse implication is true in each step. \square

This immediately implies the following corollary:

Corollary 6. *A holomorphic function space $\mathcal{HL}^2(\mathbb{C}, \alpha)$, where α is a strictly positive smooth function on \mathbb{C} , is holomorphically equivalent to one of the Segal-Bargmann spaces if and only if $\Delta \log \alpha = c < 0$. In particular, if φ is a smooth function and $\Delta \varphi$ is a positive constant, then the space $\mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$ is holomorphically equivalent to a Segal-Bargmann space.*

Proof. Note that if

$$\mu_t(z) = \frac{1}{\pi t} e^{-|z|^2/t},$$

then

$$\Delta \log \mu_t(z) = -\Delta \frac{|z|^2}{t} = -\frac{4}{t} \frac{\partial^2}{\partial z \partial \bar{z}}(z\bar{z}) = -\frac{4}{t} < 0.$$

Thus if $\mathcal{HL}^2(\mathbb{C}, \alpha)$ is holomorphically equivalent to the Segal-Bargmann space $\mathcal{HL}^2(\mathbb{C}, \mu_t)$, then $\Delta \log \alpha = \Delta \log \mu_t < 0$.

Conversely, if $\Delta \log \alpha = c < 0$, then $\Delta \log \alpha = \Delta \log \mu_t$ where $t = -4/c$. Therefore, $\mathcal{HL}^2(\mathbb{C}, \alpha)$ is holomorphically equivalent to the Segal-Bargmann space $\mathcal{HL}^2(\mathbb{C}, \mu_t)$, where $t = -4/c$. \square

3. A POINTWISE BOUND FOR A FUNCTION IN $\mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$

In this section, we obtain a pointwise bound for any function in the holomorphic function space $\mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$. First, we look at the case where $\Delta \varphi$ is a positive constant.

Theorem 7. *Let φ be a smooth function such that $\Delta\varphi = c$ where c is a positive constant. Then, for any $f \in \mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$ and any $z \in \mathbb{C}$,*

$$(3.1) \quad |f(z)|^2 \leq \frac{c}{4\pi} e^{\varphi(z)} \|f\|_{L^2(\mathbb{C}, e^{-\varphi})}^2.$$

Proof. By Corollary 6, $\mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$ is holomorphically equivalent to $\mathcal{HL}^2(\mathbb{C}, \mu_t)$, where $t = 4/c$. Then, by Lemma 3,

$$K_{e^{-\varphi}}(z, z) = \frac{1}{\pi t} e^{\varphi(z)} = \frac{c}{4\pi} e^{\varphi(z)}.$$

It follows that

$$|f(z)|^2 \leq \frac{c}{4\pi} e^{\varphi(z)} \|f\|_{L^2(\mathbb{C}, e^{-\varphi})}^2,$$

for any $f \in \mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$ and any $z \in \mathbb{C}$. \square

Note that when $\varphi = |z|^2/t$, we have $c = \Delta\varphi = 4/t$. Hence, in this case (3.1) reduces to the usual pointwise bound (1.1) for the Segal-Bargmann space.

Next, we turn to the situation in which $0 \leq \Delta\varphi \leq M$. The main result is contained in Theorem 9. But first we need to establish a technical lemma.

Recall that the function Γ defined by

$$\Gamma(z) = \frac{1}{2\pi} \log |z|$$

is the *fundamental solution* for the Laplace's equation on \mathbb{R}^2 . Thus if $\psi \in C_c^\infty(\mathbb{C})$, then

$$\Phi(z) = \Gamma * \psi(z) = \int_{\mathbb{C}} \Gamma(\zeta) \psi(z - \zeta) d\zeta$$

satisfies $\Delta\Phi = \psi$.

Lemma 8. *Let $\varphi \in C^\infty(\mathbb{C})$ satisfying $0 \leq \Delta\varphi \leq M$. Then there exists a constant C depending only on M such that for any $f \in \mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$,*

$$|f(0)|^2 \leq C e^{\varphi(0)} \int_{D(0,1)} |f(\omega)|^2 e^{-\varphi(\omega)} d\omega.$$

Proof. Choose a function $g \in C_c^\infty(\mathbb{C})$ such that $0 \leq g \leq 1$, $g = 1$ on $\overline{D(0,1)}$ and $g = 0$ outside $D(0,2)$. Let $\psi = g \Delta\varphi$. Then $\psi \in C_c^\infty(\mathbb{C})$, $0 \leq \psi \leq M$, $\psi = \Delta\varphi$ on $\overline{D(0,1)}$ and $\psi = 0$ outside $D(0,2)$. Thus $\Phi = \Gamma * \psi$ satisfies

$$(3.2) \quad \Delta\Phi(z) = \psi(z) = \Delta\varphi(z)$$

for all $z \in D(0, 1)$. First, we show that Φ is bounded above on $D(0, 1)$. Note that $\Gamma(\zeta) \leq 0$ if and only if $\zeta \in D(0, 1)$. For any $\omega \in D(0, 1)$, we have

$$\begin{aligned} \Phi(\omega) &= \int_{\mathbb{C}} \Gamma(\zeta) \psi(\omega - \zeta) d\zeta \\ &= \int_{D(\omega, 2)} \Gamma(\zeta) \psi(\omega - \zeta) d\zeta \\ &\leq \int_{D(\omega, 2) \setminus D(0, 1)} \Gamma(\zeta) \psi(\omega - \zeta) d\zeta \\ &\leq \frac{M}{2\pi} \int_{D(\omega, 2) \setminus D(0, 1)} \log |\zeta| d\zeta. \end{aligned}$$

This shows that $\Phi(\omega) \leq BM$ for all $\omega \in D(0, 1)$, where

$$B = \frac{1}{2\pi} \sup_{\omega \in D(0, 1)} \int_{D(\omega, 2) \setminus D(0, 1)} \log |\zeta| d\zeta.$$

Write $\mathcal{U} = D(0, 1)$ and let $h \in \mathcal{HL}^2(\mathcal{U}, e^{-\Phi})$. Fix $0 < s < 1$. It is not hard to show that

$$h(0) = \frac{1}{\pi s^2} \int_{D(0, s)} h(\omega) d\omega.$$

By the Cauchy-Schwarz inequality, it follows that

$$|h(0)|^2 \leq (\pi s^2)^{-2} \|\chi_{D(0, s)} e^{\Phi}\|_{L^2(\mathcal{U}, e^{-\Phi})}^2 \|h\|_{L^2(\mathcal{U}, e^{-\Phi})}^2.$$

Hence,

$$\|\chi_{D(0, s)} e^{\Phi}\|_{L^2(\mathcal{U}, e^{-\Phi})}^2 = \int_{D(0, s)} e^{\Phi(\omega)} d\omega \leq \int_{D(0, s)} e^{BM} d\omega = e^{BM} \pi s^2.$$

Thus, for any $0 < s < 1$,

$$|h(0)|^2 \leq \frac{e^{BM}}{\pi s^2} \|h\|_{L^2(\mathcal{U}, e^{-\Phi})}^2.$$

It follows that

$$|h(0)|^2 \leq \frac{e^{BM}}{\pi} \|h\|_{L^2(\mathcal{U}, e^{-\Phi})}^2$$

for all $h \in \mathcal{HL}^2(\mathcal{U}, e^{-\Phi})$. By a property of the reproducing kernel (see the paragraph preceding Definition 2) we then have

$$K_{e^{-\Phi}}(0, 0) \leq \frac{e^{BM}}{\pi}$$

where $K_{e^{-\Phi}}$ is the reproducing kernel for $\mathcal{HL}^2(\mathcal{U}, e^{-\Phi})$.

Let $K_{e^{-\varphi}}$ be the reproducing kernel for $\mathcal{HL}^2(\mathcal{U}, e^{-\varphi})$. Then, by equation (3.2) and Proposition 5, $\mathcal{HL}^2(\mathcal{U}, e^{-\varphi})$ and $\mathcal{HL}^2(\mathcal{U}, e^{-\Phi})$ are holomorphically equivalent and hence, by Lemma 3,

$$K_{e^{-\varphi}}(0, 0) = \frac{e^{-\Phi(0)}}{e^{-\varphi(0)}} K_{e^{-\Phi}}(0, 0) \leq C e^{\varphi(0)},$$

where $C = e^{BM - \Phi(0)}/\pi$. Thus

$$|h(0)|^2 \leq Ce^{\varphi(0)} \|h\|_{L^2(\mathcal{U}, e^{-\varphi})}^2,$$

for any $h \in \mathcal{HL}^2(\mathcal{U}, e^{-\varphi})$. Let $f \in \mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$ and $h = f|_{\mathcal{U}}$. Then $h \in \mathcal{HL}^2(\mathcal{U}, e^{-\varphi})$ and

$$\begin{aligned} |f(0)|^2 &= |h(0)|^2 \\ &\leq Ce^{\varphi(0)} \int_{D(0,1)} |h(\omega)|^2 e^{-\varphi(\omega)} d\omega \\ &= Ce^{\varphi(0)} \int_{D(0,1)} |f(\omega)|^2 e^{-\varphi(\omega)} d\omega. \end{aligned}$$

Finally, it remains to show that we can choose a constant C to depend only on M . By straightforward calculations, we have

$$\int_{D(0,1)} \Gamma(\zeta) d\zeta = -\frac{1}{4}.$$

Now, consider

$$\Phi(0) = \int_{\mathbb{C}} \Gamma(\zeta) \psi(-\zeta) d\zeta \geq \int_{D(0,1)} \Gamma(\zeta) \psi(-\zeta) d\zeta \geq -\frac{M}{4}.$$

Thus $e^{-\Phi(0)} \leq e^{\frac{M}{4}}$, which shows that $C \leq \frac{1}{\pi} e^{(B+\frac{1}{4})M}$. \square

Theorem 9. *Let $\varphi \in C^\infty(\mathbb{C})$ with $0 \leq \Delta\varphi \leq M$. Then there exists a constant C depending only on M such that for any $f \in \mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$ and any $z \in \mathbb{C}$,*

$$|f(z)|^2 \leq Ce^{\varphi(z)} \|f\|_{L^2(\mathbb{C}, e^{-\varphi})}^2.$$

Proof. Let $z \in \mathbb{C}$ and $g_z(\omega) = z + \omega$. Then $0 \leq \Delta(\varphi \circ g_z) \leq M$. Let $f \in \mathcal{HL}^2(\mathbb{C}, e^{-\varphi})$ and $h = f \circ g_z$. Then $h \in \mathcal{HL}^2(\mathbb{C}, e^{-\varphi \circ g_z})$ and by Lemma 8,

$$\begin{aligned} |f(z)|^2 &= |f \circ g_z(0)|^2 = |h(0)|^2 \\ &\leq Ce^{\varphi \circ g_z(0)} \int_{D(0,1)} |h(\omega)|^2 e^{-\varphi \circ g_z(\omega)} d\omega \\ &= Ce^{\varphi(z)} \int_{D(0,1)} |f \circ g_z(\omega)|^2 e^{-\varphi \circ g_z(\omega)} d\omega \\ &= Ce^{\varphi(z)} \int_{D(0,1)} |f(z + \omega)|^2 e^{-\varphi(z + \omega)} d\omega \\ &\leq Ce^{\varphi(z)} \int_{\mathbb{C}} |f(\omega)|^2 e^{-\varphi(\omega)} d\omega \\ &= Ce^{\varphi(z)} \|f\|_{L^2(\mathbb{C}, e^{-\varphi})}^2. \end{aligned}$$

\square

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